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MATRIX FACTORIZATION METHOD IN MIXED STATIC PROBLEMS OF AN ELASTIC WEDGE

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We study the plane contact static problems of an elastic wedge under the condition that it is rigidly connected with the stamp. The question of solvability of the above problems is investigated and an approximate method of solution involving the matrix factorization method developed in paper [1] and others, is proposed. The plane problems of elasticity for a wedge with discontinuous boundary conditions were investigated by a number of authors. For example, in [2] the author used the method of reduction to an integral Wiener-Hopf equation to investigate the problem of indenting a rigid stamp into a perfectly smooth face of an elastic wedge. In [3] a similar problem was reduced to a certain Fredholm equation of second kind. In [4] and others (*), the asymptotic and orthogonal polynomial methods were successfully used.

1. We shall consider an elastic wedge, the upper face of which is acted upon by a strip stamp rigidly adhering to the face. The boundaries of the zone of contact are separated from the edge of the wedge by the distances a^* and b^*, respectively $(b^* > a^* > 0)$, and we study the case of plane deformation. For a displacement vector $\mathbf{u}^\circ(r)=\{u_r^\circ(r), u_\varphi^\circ(r)\}$ defined in the region $\varphi = \alpha,\ a^* < r < b^*$, we require to find the total stress vector $\sigma^\circ(r) = \{\tau_{\varphi r}^\circ(r), \sigma_\varphi^\circ(r)\}$ in the zone of contact for each of the following conditions of clamping of the lower edge of the wedge $(\varphi = 0,\ 0 < r < \infty)$:

$$\text{A}\quad u_r(r, 0) = u_\varphi(r, 0) = 0$$

$$\text{B}\quad \tau_{\varphi r}(r, 0) = \sigma_\varphi(r, 0) = 0$$

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*) Lutchenko, S. A. and Popov, G. Ia. On certain plane contact problems of the theory of elasticity for a wedge. In coll.: 3-rd All-Union Convention on Theoretical and Applied Mechanics. Moscow, 1968. Annot. dokl., Moscow, "Nauka", 1968.

$$\text{C} \quad \tau_{\varphi r}(r, 0) = u_{\varphi}(r, 0) = 0$$

Here r and φ are the polar coordinates in any cross section of the wedge perpendicular to the edge. We also assume that all stresses and displacements vanish as $r \to \infty$ on any ray $\varphi = \text{const}$.

We shall utilize the general solution of the plane theory of elasticity in the form of Papkovich-Neuber, as given in [5]. Satisfying the boundary conditions (1.1), we arrive at a system of integral equations for the unknown vector of contact stresses $\boldsymbol{\sigma}^{\circ}(r)$ which can be written in the matrix form as

$$\mathbf{L}\mathbf{s} = \int_a^b \mathbf{k}\left(\frac{\rho}{\rho'}\right)\mathbf{s}(\rho')\,d\rho' = \mathbf{f}(\rho), \quad a \leqslant \rho \leqslant b \tag{1.2}$$

$$\mathbf{k}(t) = \frac{1}{2\pi}\int_{-\infty+i\varepsilon}^{-\infty+i\varepsilon} \mathbf{K}(z)\,t^{-iz/2\alpha}dz, \quad r = h\rho, \quad h = b^* - a^*, \quad \varepsilon > 0$$

$$\mathbf{K}(z) = \left\| \begin{matrix} K_{11}(z) & iK_{12}(z) \\ -iK_{21}(z) & K_{22}(z) \end{matrix} \right\|, \quad a^* = ha, \quad b^* = hb$$

$$\mathbf{s}(\rho) = \left\| \begin{matrix} s_1 \\ s_2 \end{matrix} \right\| = \left\| \begin{matrix} \tau_{\varphi r}^{\circ}(\rho)/2G \\ \sigma_{\varphi}^{\circ}(\rho)/2G \end{matrix} \right\|, \quad \mathbf{f}(\rho) = \left\| \begin{matrix} f_1 \\ f_2 \end{matrix} \right\| = \left\| \begin{matrix} u_r^{\circ}(\rho)/h \\ u_{\varphi}^{\circ}(\rho)/h \end{matrix} \right\|$$

(G is the shear modulus of the wedge material). In each of these cases A, B and C of the boundary conditions at the lower edge of the wedge the expressions for the elements of the matrix $\mathbf{K}(z)$, are determined by the following respective relations:

$$\text{A} \quad \Delta(z)K_{11}(z) = C\left(\varkappa\frac{\operatorname{sh} z}{z} + \frac{\sin 2\alpha}{2\alpha}\right) \tag{1.3}$$

$$\Delta(z)K_{22}(z) = C\left(\varkappa\frac{\operatorname{sh} z}{z} - \frac{\sin 2\alpha}{2\alpha}\right)$$

$$\Delta(z)K_{12}(z) = \frac{z}{2}\left(B\frac{\operatorname{sh}^2 {}^1\!/_2 z}{{}^1\!/_4 z^2} + \frac{\sin^2\alpha}{\alpha^2}\right) + iC\frac{\sin^2\alpha}{\alpha}$$

$$\Delta(z) = \varkappa\operatorname{ch} z + \frac{\sin^2\alpha}{2\alpha^2}z + C^2, \quad K_{21}(z) = \overline{K_{12}(z)}$$

$$\text{B} \quad \Delta(z)K_{11}(z) = C\left(\frac{\operatorname{sh} z}{z} + \frac{\sin 2\alpha}{2\alpha}\right)$$

$$\Delta(z)K_{22}(z) = C\left(\frac{\operatorname{sh} z}{z} - \frac{\sin 2\alpha}{2\alpha}\right)$$

$$\Delta(z)K_{12}(z) = \frac{z}{2}\left(B\frac{\operatorname{sh}^2 {}^1\!/_2 z}{{}^1\!/_4 z^2} + \frac{\sin^2\alpha}{\alpha^2}\right) + iC\frac{\sin^2\alpha}{\alpha}$$

$$\Delta(z) = \frac{z^2}{2}\left(\frac{\operatorname{sh}^2 {}^1\!/_2 z}{{}^1\!/_4 z^2} - \frac{\sin^2\alpha}{\alpha^2}\right), \quad K_{21}(z) = \overline{K_{12}(z)}$$

$$\text{C}\,\Delta(z)K_{11}(z) = C(\operatorname{ch} z + \cos 2\alpha)$$

$$\Delta(z)K_{22}(z) = C(\operatorname{ch} z - \cos 2\alpha)$$

$$\Delta(z)K_{12}(z) = z\left(B\frac{\operatorname{sh} z}{z} - \frac{\sin 2\alpha}{2\alpha}\right) + iC\sin 2\alpha$$

$$\Delta(z) = z^2\left(\frac{\operatorname{sh} z}{z} + \frac{\sin 2\alpha}{2\alpha}\right), \quad K_{21}(z) = \overline{K_{12}(z)}$$

$$B = 1 - 2\nu, \qquad C = 2(1 - \nu)$$

where a bar superscript denotes the complex conjugate and ν is the Poisson's ratio. It can easily by shown that the asymptotic expressions for the functions $K_{ij}(z)$ as $|z| \to \infty$, are given by

$$K_{ii}(z) = C\,|z|^{-1}[1 + O(z^2 \operatorname{ch}^{-1} z)] \tag{1.4}$$

$$K_{ij}(z) = B\,|z|^{-1} \operatorname{sgn} z\,[1 + O(z^2 \operatorname{ch}^{-1} z)]$$

Lemma 1. The following asymptotic estimates hold:

$$k_{ii}(\rho/\rho') = C_1 \ln\tau\,[1 + o(\tau)], \quad k_{ij}(\rho/\rho') = O(1) \tag{1.5}$$

$$\tau = |\ln(\rho/\rho')| \to 0$$

(k_{ij} are the elements of the matrix $\mathbf{k}(t)$) .

2. Let us investigate the question of solvability of the system (1.2). To do this, we consider the set of vector functions $\mathrm{s}(\rho)$ for which the following functional exists:

$$E(\mathrm{s}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\overline{\mathrm{s}^*(z)}\cdot\mathbf{K}(z)\cdot\mathrm{s}^*(z)\}\,dz < \infty \tag{2.1}$$

$$\mathrm{s}^*(z) = \int_a^b \mathrm{s}(\rho)\,\rho^{-iz/2\alpha}\,d\rho$$

The matrix $\mathbf{K}(z)$ has, for each of the problems A, B and C, a concrete form determined by the relations (1.3). When considering the problems B and C we shall require that the conditions

$$\mathrm{s}^*(0) = 0 \tag{2.2}$$

are satisfied.

Lemma 2. We have the inequality $E(\mathrm{s}) \geqslant 0$; the equality is attained only when $\mathrm{s}(\rho) \equiv 0$.

To prove this, we must reduce the functional (2.1) to the form

$$E(\mathrm{s}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e(s_1^*, s_2^*)\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{K_{11}(z)\,|s_1^*(z)|^2 + \tag{2.3}$$
$$K_{22}(z)\,|s_2^*(z)|^2 + 2\operatorname{Re}[K_{12}(z)\overline{s_1^*(z)}\,s_2^*(z)]\}\,dz$$

Then the obvious inequality

$$E(\mathrm{s}) \geqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x_1, x_2)\,dz \tag{2.4}$$

$$T(x_1, x_2) = K_{11}(z)\,x_1^2 + K_{22}(z)\,x_2^2 - 2\,|K_{12}(z)|\,x_1 x_2$$
$$x_1 = |s_1^*(z)|, \quad x_2 = |s_2^*(z)|$$

and the positive definiteness of the quadratic form $T(x_1, x_2)$ provide the proof of the lemma.

The above lemma enables us to introduce the space $H(a, b)$ of generalized solutions of the system (1.2) with the norm $\|\mathrm{s}\|_H^2 = E(\mathrm{s})$ defined by the right-hand side of the relation (2.3).

Lemma 3. The inequality

$$m(z)\,|\,\mathbf{s}^*(z)\,| \leqslant e(s_1^*,\ s_2^*) \leqslant M(z)\,|\mathbf{s}^*(z)\,| \qquad (2.5)$$
$$0 < m(z) < M(z) < \infty, \quad m(z),\ M(z) = O(z^{-1})$$

(where the explicit expressions for m and M are easily obtained), holds. The proof can be reduced to showing that the maximum and minimum of the form $e(s_1^*, s_2^*)$ both exist on the unit circle.

Theorem 1. The integral equations of the problems A, B and C have unique generalized solutions in the space $H(a, b)$ provided that

$$\int_{-\infty}^{\infty} |\,\mathbf{F}(z)|^2 m^{-1}(z)\,dz < \infty, \quad \mathbf{F}(z) = \int_0^{\infty} \mathbf{f}_{ab}(\rho)\,\rho^{-iz/2\alpha} d\rho \qquad (2.6)$$

where $\mathbf{f}_{ab}(\rho)$ is the continuation of $\mathbf{f}(\rho)$ from the segment $a \leqslant \rho \leqslant b$ to the whole of the semiaxis $0 < \rho < \infty$, carried out with the help of the left-hand side of (1.2).

To prove the theorem we must introduce a Hilbert space of generalized solutions with the scalar product of the elements of $H(a, b)$ generating the norm (2.3), and use the Riesz theorem with inequality (2.5).

We shall now establish that the class of uniqueness contains the class of summable functions. In addition to $H(a, b)$ we introduce the spaces $C(a, b)$ and $L_p(a, b)$, as well as $p > 1$ vector functions $\mathbf{s}(\rho) = \{s_1(\rho), s_2(\rho)\}$ defined by the usual metric

$$\|\mathbf{s}\|_C = \max_n \sup_\rho |\,s_n(\rho)\,|, \quad a \leqslant \rho \leqslant b,\ n = 1, 2 \qquad (2.7)$$
$$\|\mathbf{s}\|_{L_p} = \Big(\int_a^b |\,\mathbf{s}(\rho)\,|^p\,d\rho\Big)^{1/p}, \quad p > 1$$

Using arguments similar to those used in [6], we can establish the following theorems:

Theorem 2. The operator $\mathbf{L}$ in the relations (1.2) acts from $L_p(a, b)$ to $C(a, b)$ continuously.

Theorem 3. The imbedding

$$L_p(a, b) \subset H(a, b), \quad 1 < p \leqslant 2$$

is valid.

Proof of the above theorems follows directly from the inequalities (2.5) and the Hausdorff-Young inequality for the Fourier integrals.

From the results of Theorems 2 and 3 follows

Theorem 4. Equation (1.2) cannot have more than one solution in the space $L_p(a, b)$, $1 < p \leqslant 2$

Note. Lemma 2 establishes the positive definiteness of the operator $\mathbf{L}$ appearing in the left-hand side of (1.2), and this in fact means the non-negativeness of the energy functional $E(\mathbf{s})$ accumulated by the elastic medium during its deformation. The condition (2.2) ensures that the energy in the problems B and C is finite, and represents the condition of self-balancing of the loading at the upper face of the wedge.

3. We continue our investigations by reducing Eq. (1.2) to an equivalent system of the Fredholm integral equations of second kind. We proceed from the relation

$$\mathbf{K}(z)\,\mathbf{s}^*(z) = \mathbf{f}^*(z) + \mathbf{f}_a^*(z) + \mathbf{f}_b^*(z), \quad -\infty < z < \infty \qquad (3.1)$$

which follows from (1.2). We denote by $\mathbf{f}_a^*(z)$ and $\mathbf{f}_b^*(z)$ the Mellin transforms of the natural continuation of $\mathbf{f}(\rho)$ into the regions $0 \leqslant \rho \leqslant a$ and $a \leqslant \rho < b$, respectively. It can easily be established that the vector functions

$$\mathbf{\Phi}^+(z) = \mathbf{f}_a^*(z)\, a^{-iz/2\alpha}, \quad \mathbf{F}^+(z) = \mathbf{f}^*(z)\, b^{-iz/2\alpha} \tag{3.2}$$

are regular in the upper semiplane and vanish as $\operatorname{Im} z \to +\infty$, while the vector functions

$$\mathbf{\Psi}^-(z) = \mathbf{f}_b^*(z)\, b^{-iz/2\alpha}, \quad \mathbf{F}^-(z) = \mathbf{f}^*(z)\, a^{-iz/2\alpha} \tag{3.3}$$

are regular in the lower semiplane and vanish as $\operatorname{Im} z \to -\infty$. Then by virtue of (3.2) and (3.3) the relation (3.1) can be written in the following two equivalent forms:

$$\mathbf{K}(z)\,\mathbf{S}^+(z) = \mathbf{F}^+(z) + \mathbf{\Phi}^+(z)\,\lambda^{iz/2\alpha} + \mathbf{\Psi}^-(z) \tag{3.4}$$

$$\mathbf{K}(z)\,\mathbf{S}^-(z) = \mathbf{F}^-(z) + \mathbf{\Phi}^+(z) + \mathbf{\Psi}^-(z)\,\lambda^{-iz/2\alpha}$$
$$-\infty < z < \infty, \quad \lambda = b/a > 1$$

In order to apply the factorization method to the relations (3.4), we must factorize the matrix-function $\mathbf{K}(z)$ relative to the real axis in the complex z-plane. From the asymptotic relations (1.4) it follows that when $|z| \to \infty$, the matrix is close to a certain functional-commutative matrix [1] factorized in a finite form.

Taking all this into account, we introduce the following functional-commutative matrix:

$$\mathbf{A}(z) = \begin{Vmatrix} K_{11}(z) & iK_{12}(z) \\ -iK_{12}(z) & K_{11}(z) \end{Vmatrix}, \quad -\infty < z < \infty \tag{3.5}$$

On the basis of the theory of matrix functions [7] we can obtain the factorization formulas in the form

$$\mathbf{A}(z) = \mathbf{A}_+(z)\mathbf{A}_-(z), \quad R_+(z)\,R_-(z) = K_{11}(z) + K_{12}(z) \tag{3.6}$$
$$K_+(z)K_-(z) = K_{11}(z) - K_{12}(z)$$

$$\mathbf{A}_\pm(z) = \begin{Vmatrix} R_{11}^\pm(z) & iR_{12}^\pm(z) \\ -iR_{12}^\pm(z) & R_{11}^\pm(z) \end{Vmatrix}, \quad \begin{aligned} R_{11}^\pm(z) &= R_\pm(z) + K_\pm(z) \\ R_{12}^\pm(z) &= R_\pm(z) - K_\pm(z) \end{aligned}$$

Lemma 4. The following estimates are valid [1]:

$$R_\pm(z) = C_1(\mp iz)^{-\tau_\mp}[1 + o(z^{-1})], \quad \tau_\pm = {}^1/_2\,(1 \pm i\pi^{-1}\ln \varkappa) \tag{3.7}$$
$$K_\pm(z) = C_2(\mp iz)^{-\tau_\pm}[1 + o(z^{-1})], \quad \varkappa = 3 - 4\nu$$

The proof follows from the estimates (1.4) and relations (3.6). We can now represent the matrix $\mathbf{K}(z)$ in its two equivalent forms

$$\mathbf{K}(z) = \mathbf{A}_-(z)\,\mathbf{\Pi}(z)\,\mathbf{A}_+(z), \quad \mathbf{K}(z) = \mathbf{A}_+(z)\,\mathbf{\Lambda}(z)\,\mathbf{A}_-(z) \tag{3.8}$$

Theorem 5. The elements of the matrix $\mathbf{\Pi}(z)$ have the form

$$\Pi_{11}(z) = 1 + \frac{1}{4}\,[R_{12}^- R_{12}^+(K_{22} - K_{11}) + R_{11}^+ R_{12}^-(K_{12} - \overline{K}_{12})]\,\theta^{-1} \tag{3.9}$$

$$\Pi_{22}(z) = 1 + \frac{1}{4}\,[R_{11}^- R_{11}^+(K_{22} - K_{11}) + R_{11}^+ R_{12}^-(K_{12} - \overline{K}_{12})]\,\theta^{-1}$$

$$\Pi_{12}(z) = -\frac{i}{4}\,[R_{12}^- R_{11}^+(K_{22} - K_{11}) + R_{12}^- R_{12}^+(K_{12} - \overline{K}_{12})]\,\theta^{-1}$$

$$\Pi_{21}(z) = \frac{i}{4}[R_{11}^{-}R_{12}^{+}(K_{22}-K_{11}) + R_{11}^{-}R_{11}^{+}(K_{12}-\overline{K}_{12})]\theta^{-1}$$
$$\theta = \det \mathbf{K}(z)$$

and the following asymptotic properties:

$$\Pi_{11}(z),\ \Pi_{22}(z) = 1 + O(z^2 \mathrm{ch}^{-1} z) \tag{3.10}$$
$$\Pi_{12}(z),\ \Pi_{21}(z) = O(z^2 \mathrm{ch}^{-1} z), \quad |z| \to \infty$$

The proof of the theorem is based on indirect computation of the elements of the matrix $\mathbf{\Pi}(z)$ and the application of the asymptotic estimates (3.7). Relations similar to (3.9) and (3.10) appear to be valid also for the matrix $\mathbf{\Lambda}(z)$.

By virtue of the asymptotic estimates of the form (3.10), the elements of the matrices $\mathbf{\Pi}(z)$ and $\mathbf{\Lambda}(z)$ can be approximated with any degree of accuracy using the bilinear functions. The resulting matrices can then be factorized using the results [8] of factorization of the matrix-functions, yielding the representations of the form

$$\mathbf{\Pi}(z) \approx \mathbf{\Pi}^*(z)\mathbf{\Pi}_+^*(z), \quad \mathbf{\Lambda}(z) \approx \mathbf{\Lambda}_+^*(z)\mathbf{\Lambda}_-^*(z) \tag{3.11}$$

where $\mathbf{\Pi}_\pm^*(z)$ and $\Lambda_\pm^*(z)$ are matrix functions with bilinear elements, regular in the regions $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively. Introducing the expressions (3.8) into (3.4) and performing the factorization relative to the real axis with (3.11) taken into account, we arrive at the relations

$$\{\mathbf{U}_-^{-1}(z)\mathbf{F}^+(z)\}^- + \{\mathbf{U}_-^{-1}(z)\mathbf{\Phi}^+(z)\lambda^{iz/2\alpha}\}^- + \mathbf{U}_-^{-1}(z)\mathbf{\Psi}^-(z) = 0, \tag{3.12}$$
$$\mathbf{U}_-(z) = \mathbf{A}_-(z)\mathbf{\Pi}_-^*(z)$$
$$\{\mathbf{V}_+^{-1}(z)\mathbf{F}^-(z)\}^+ + \{\mathbf{V}_+^{-1}(z)\mathbf{\Psi}^-(z)\lambda^{-iz/2\alpha}\}^+ + \mathbf{V}_+^{-1}(z)\mathbf{\Phi}^+(z) = 0,$$
$$\mathbf{V}_+(z) = \mathbf{A}_+(z)\mathbf{\Lambda}_+^*(z)$$

Let us introduce the following notation:

$$\mathbf{U}_-^{-1}(z)\mathbf{V}_+(z) = \mathbf{C}(z) \tag{3.13}$$
$$\mathbf{U}_-^{-1}(z)\mathbf{\Psi}^-(z) = -\mathbf{X}_1(z), \quad \mathbf{V}_+^{-1}(z)\mathbf{\Phi}^+(z) = \mathbf{X}_2(z)$$

By virtue of the relations (3.12) and (3.13) and the factorization theorems we arrive at the following system of integral equations of second kind, relative to the unknown vectors $\mathbf{X}_1(z)$ and $\mathbf{X}_2(z)$ $(\operatorname{Im} z < 0)$:

$$2\pi i\mathbf{X}_1(z) = \int_\Gamma \frac{\mathbf{C}(-\zeta)\mathbf{X}_2(\zeta)}{\zeta+z}\lambda^{-i\zeta/2\alpha}d\zeta + \int_\Gamma \frac{\mathbf{U}_-^{-1}(\zeta)\mathbf{F}^+(\zeta)}{\zeta+z}\lambda^{-i\zeta/2\alpha}d\zeta \tag{3.14}$$
$$2\pi i\mathbf{X}_2(z) = \int_\Gamma \frac{\mathbf{C}^{-1}(\zeta)\mathbf{X}_2(\zeta)}{\zeta+z}\lambda^{-i\zeta/2\alpha}d\zeta + \int_\Gamma \frac{\mathbf{V}_+^{-1}(\zeta)\mathbf{F}^-(\zeta)}{\zeta+z}\lambda^{-i\zeta/2\alpha}d\zeta$$

where the contour Γ is situated in the half-plane $\operatorname{Im}\zeta < 0$ and encloses the zeros of the integrand functions on this half-plane from above. Then, in the case when the system (3.14) has a solution, the relations (3.4) yield an expression for the stress vector in the regions of contact, of the form

$$2\pi s(\rho)\rho = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathbf{K}^{-1}(u)\mathbf{f}^*(u)\rho^{-iu/2\alpha}du - \tag{3.15}$$

$$\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathbf{U}_+^{-1}(u)\mathbf{X}_1(u)\left(\frac{\rho}{a}\right)^{-iu/2\alpha}du + \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathbf{V}_+^{-1}(u)\mathbf{X}_2(-u)\left(\frac{b}{\rho}\right)^{-iu/2\alpha}du$$

$$s(\rho) = \sigma^\circ(r)/2G$$

To investigate the problem of solvability of (3.14), we introduce the space $c(\gamma)$ of vector functions $\mathbf{f}(z) = \{f_1(z), f_2(z)\}$, $z \in \Gamma$ defined by the metric

$$\|\mathbf{f}\|_{c(\gamma)} = \max_n \max_z |f_n(z)z^\gamma|, \lim |f_n(z)z^\gamma| = 0, \quad |z| \to \infty, \quad \gamma > 0$$

Theorem 6. If the free term of the system (3.14) belongs to the space $c(\gamma)$, $0 < \gamma < 1$, then the system has a unique solution in this space.

To prove this theorem we establish the complete continuity of the operator appearing in the left-hand side of (3.14) in $c(\gamma)$, $0 < \gamma < 1$. The contour of integration Γ in the lower half-plane is predeformed in such a way, that its sufficiently distant segments lie on the bisectrices of the angles of the third and fourth quadrant of the ζ-coordinate plane, and the validity of the conditions of the compactness criterion is then verified.

4. As an example, we examine in more detail the problem of a stamp with a plane base, rigidly adhering to an elastic wedge, the lower face of which is also rigidly fixed (problem A).

In accordance with the theory stated above and the relation (1.3) A taken into account, it was found convenient to factorize the functions (3.6) with the help of the following approximation:

$$K_{11}(z) \pm K_{12}(z) \approx \sqrt{\varkappa}\, b^{\pm} \frac{\operatorname{sh}(b^{\pm}z + {}^1\!/_2 \ln \varkappa)}{b^{\pm}z + {}^1\!/_2 \ln \varkappa} \frac{P_{2n}(z)}{Q_{2n}(z)} \tag{4.1}$$

Here $P_{2n}(z)$ and $Q_{2n}(z)$ are polynomials of $2n$-th degree, and $b^{\pm}$ are the constants of the approximation obtained from the conditions that the left and right-hand sides become equal when $z = 0$. (Factorization of the expression appearing in the right-hand side presents no difficulty, and can be carried out in an exact manner relative to the real axis. In addition, the estimates (3.7) hold).

In the course of constructing the approximate solution, we have limited ourselves, in the present case, to $n = 0$. Numerical analysis shows that in this case the error of the approximation (4.1) does not exceed 2%.

In order to factorize approximately the matrices $\Pi(z)$ and $\Lambda(z)$, the elements of these matrices were previously approximated in accordance with (3.8), using bilinear functions and the Bernshtein polynomials [1, 9]. The resulting system of second kind (3.14) was then reduced to an infinite system by expanding the integrals appearing in it, into series in terms of the residues. The asymptotic solution of the system was constructed under the assumption that $\lambda = b/a > 1$.

Having found the vector functions $\mathbf{X}_{1,2}(z)$, we obtain the required contact stresses, using the formulas (3.15). The integrals appearing as the result, are converted into the formulas of the operational calculus and computed in their closed form [10]. The principal term of the asymptotic expression for the contact stress vector has the following form ($\lambda \gg 1$):

$$Y\rho\,[\tau_{\varphi r}^{*}(\rho) + i\sigma_{\varphi}^{\circ}(\rho)]\,/\,4G = -\,\mathbf{K}^{-1}(0)\,\mathbf{f}(\rho) - \tag{4.2}$$
$$\{Y_a\Gamma^{-1}(\tau_+)\,(\rho/a)^{-\beta\varepsilon_-}\,[1-(\rho/a)^{-\beta}]^{-\tau_+} +$$
$$Y_b\Gamma^{-1}(\tau_-)\,(b/\rho)^{-\beta\varepsilon_+}\,[1-(b/\rho)^{-\beta}]^{-\tau_-}\}\,[1+O(\alpha^{-1}\lambda^{-m/2\alpha})]$$
$$\beta = \pi/2\alpha b^{+},\ \tau_{\pm} = {}^{1}/_{2}\,(1 \pm i\pi^{-1}\ln\varkappa),\ \varepsilon_{\pm} = \tau_{\pm} - {}^{1}/_{2},\ m > 0$$

where $\Gamma(z)$ is the Eulerian integral of second kind, $\mathbf{f}(\rho)$ is the displacement vector in the region of contact and Y, Y_a and Y_b depend on the constants of the approximation.

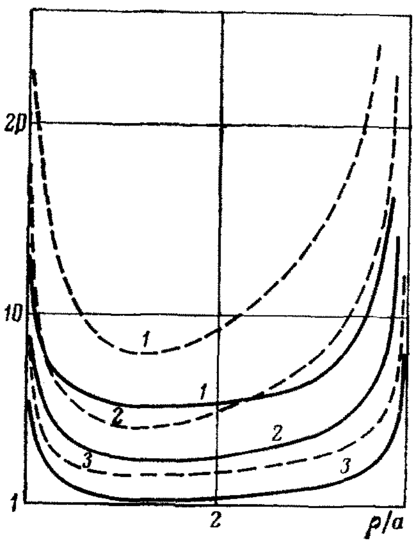

Fig. 1

For the purpose of numerical analysis it was set $\mathbf{f}(\rho) = \{1, 2\}$, $\varkappa = 1.6$ and $\lambda = 3$. Figure 1 shows the dependence on ρ/a of $\alpha\tau_{\varphi r}^{\circ}/2G$ (solid lines) and $\alpha\sigma_{\varphi}^{\circ}/2G$ (broken lines). The curves 1, 2 and 3 correspond to the values of the wedge angle α of $3\pi/10$, $\pi/5$ and $\pi/10$. The contact stress curves are distributed in the reverse order with increasing α.

Note. The relation (4.2) implies that a considerable oscillation of the contact stresses is observed in the neighborhood of the zone of contact boundaries. The frequency of these oscillations increases on approaching the boundaries $\rho = a$ and $\rho = b$. A similar phenomenon was noted earlier by the authors of [11 – 13] and others in the course of investigating the contact problems with the coupling forces taken into account. In fact, the above effect is not observed. It arises as the result of the breakdown of the linearity of the relations connecting the stresses and deformations during the penetration of the sharp edge of the stamp rigidly coupled to the medium. Analysis of the relations (4.2) shows that the oscillation appears at very small distances from the stamp edges (about 10^{-4} of the width of the contact zone) and it is therefore not shown on Fig. 1.

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SOME INVERSE PROBLEMS FOR FLEXIBLE PLATES

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We investigate the problems of finding the optimal loads acting on a plate, which insure the best root-mean-square (RMS) approximation to a given distribution of bending and torsional moments, or of the displacements. We study the problems of existence and uniqueness of the optimal solution, and establish the necessary and sufficient conditions of optimality under the assumption that the manifold of admissible loads is a closed convex set in some Hilbert space.

1. Certain relationships of the theory of plates. Auxiliary assumptions. We shall consider the inverse problems of plates of variable thickness. The equation of flexure of such a plate has the form [1]

$$
\begin{gathered}
Pu = \frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}\right)\right] + \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial x^2}\right)\right] + \\
2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D\frac{\partial^2 u}{\partial x\,\partial y}\right) = g, \quad (x, y) \in \Omega
\end{gathered}
\tag{1.1}
$$

Here $u(x, y)$ denotes the deflection of the median plane of the plate, ν is the Poisson's ratio which is a nonnegative constant, $g(x, y)$ is the external load intensity, $D(x, y)$ is the torsional rigidity of the plate and Ω is an open bounded region on the